

# Compiling Finite Linear CSP into SAT

Naoyuki Tamura<sup>1</sup>, Akiko Taga<sup>2</sup>, Satoshi Kitagawa<sup>2</sup>, and Mutsunori Banbara<sup>1</sup>

<sup>1</sup> Information Science and Technology Center, Kobe University, JAPAN  
tamura@kobe-u.ac.jp

<sup>2</sup> Graduate School of Science and Technology, Kobe University, JAPAN

**Abstract.** In this paper, we propose a method to encode Constraint Satisfaction Problems (CSP) and Constraint Optimization Problems (COP) with integer linear constraints into Boolean Satisfiability Testing Problems (SAT). The encoding method is basically the same with the one used to encode Job-Shop Scheduling Problems by Crawford and Baker. Comparison  $x \leq a$  is encoded by a different Boolean variable for each integer variable  $x$  and integer value  $a$ . To evaluate the effectiveness of this approach, we applied the method to Open-Shop Scheduling Problems (OSS). All 192 instances in three OSS benchmark sets are examined, and our program found and proved the optimal results for all instances including three previously undecided problems.

## 1 Introduction

Recent advances in SAT solver technologies [1–5] have enabled solving a problem by encoding it to a SAT problem, and then to use the efficient SAT solver to find a solution, such as for model checking, planning, and scheduling [6–12].

In this paper, we propose a method to encode Constraint Satisfaction Problems (CSP) and Constraint Optimization Problems (COP) with integer linear constraints into Boolean Satisfiability Testing Problems (SAT) of CNF (product-of-sums) formulas.

As Hoos discussed in [8], basically two encoding methods are known: “sparse encoding” and “compact encoding”. Sparse encoding [13] encodes each assignment of a value to an integer variable by a different Boolean variable, that is, Boolean variable representing  $x = a$  is used for each integer variable  $x$  and integer value  $a$ . Compact encoding [14, 7] assigns a Boolean variable for each bit of each integer variable.

Encoding method used in this paper is different from these. The method is basically the same with the one used to encode Job-Shop Scheduling Problems by Crawford and Baker in [9] and studied by Soh, Inoue, and Nabeshima in [10–12]. It encodes a comparison  $x \leq a$  by a different Boolean variable for each integer variable  $x$  and integer value  $a$ .

The benefit of this encoding is the natural representation of the order relation on integers. Axiom clauses with two literals, such as  $\{\neg(x \leq a), x \leq a + 1\}$  for each integer  $a$ , represent the order relation for an integer variable  $x$ . Clauses,

for example  $\{x \leq a, \neg(y \leq a)\}$  for each integer  $a$ , can be used to represent the constraint among integer variables, i.e.  $x \leq y$ .

The original encoding method in [9–12] is only for Job-Shop Scheduling Problems. In this paper, we extend the method so that it can be applied for any finite linear CSPs and COPs.

To evaluate the effectiveness of this approach, we applied the method to Open-Shop Scheduling Problems (OSS). All 192 instances in three OSS benchmark sets [15–17] are examined, and our program found and proved the optimal results for all instances including three previously undecided problems [18–20].

## 2 Finite Linear CSP and SAT

In this section, we define finite linear *Constraint Satisfaction Problems* (CSP) and *Boolean Satisfiability Testing Problems* (SAT) of CNF formulas.

$\mathbf{Z}$  is used to denote a set of integers and  $\mathbf{B}$  is used to denote a set of Boolean constants ( $\top$  and  $\perp$  are the only elements of  $\mathbf{B}$  representing “true” and “false” respectively).

We also prepare two countably infinite sets of *integer variables*  $\mathcal{V}$  and *Boolean variables*  $\mathcal{B}$ . Although only a finite number of variables are used in a specific CSP or SAT, countably infinite variables are prepared to introduce new variables during the translation. Symbols  $x, y, z, x_1, y_1, z_1, \dots$ , are used to denote integer variables, and symbols  $p, q, r, p_1, q_1, r_1, \dots$ , are used to denote Boolean variables.

*Linear expressions* over  $V \subset \mathcal{V}$ , denoted by  $E(V)$ , are algebraic expressions in the form of  $\sum a_i x_i$  where  $a_i$ 's are non-zero integers and  $x_i$ 's are integer variables (elements of  $V$ ). We also add the restriction that  $x_i$ 's are mutually distinct.

*Literals* over  $V \subset \mathcal{V}$  and  $B \subset \mathcal{B}$ , denoted by  $L(V, B)$ , consist of Boolean variables  $\{p \mid p \in B\}$ , negations of Boolean variables  $\{\neg p \mid p \in B\}$ , and *comparisons*  $\{e \leq c \mid e \in E(V), c \in \mathbf{Z}\}$ . Please note that we restrict comparison literals to only appear positively and in the form of  $\sum a_i x_i \leq c$  without loss of generality. For example,  $\neg(a_1 x_1 + a_2 x_2 \leq c)$  can be represented with  $-a_1 x_1 - a_2 x_2 \leq -c - 1$ , and  $x \neq y$  (that is,  $(x < y) \vee (x > y)$ ) can be represented with  $(x - y \leq -1) \vee (-x + y \leq -1)$ .

*Clauses* over  $V \subset \mathcal{V}$  and  $B \subset \mathcal{B}$ , denoted by  $C(V, B)$ , are defined as usual where literals are chosen from  $L(V, B)$ , that is, a clause represents a disjunction of element literals. Integer variables occurring in a clause are treated as free variables, that is, a clause  $\{x \leq 0\}$  does not mean  $\forall x.(x \leq 0)$ .

**Definition 1 (Finite linear CSP).** A (finite linear) CSP (Constraint Satisfaction Problem) is defined as a tuple  $(V, \ell, u, B, S)$  where

- (1)  $V$  is a finite subset of *integer variables*  $\mathcal{V}$ ,
- (2)  $\ell$  is a mapping from  $V$  to  $\mathbf{Z}$  representing the *lower bound* of the integer variable,
- (3)  $u$  is a mapping from  $V$  to  $\mathbf{Z}$  representing the *upper bound* of the integer variable,
- (4)  $B$  is a finite subset of *Boolean variables*  $\mathcal{B}$ , and

- (5)  $S$  is a finite set of clauses (that is, a finite subset of  $C(V, B)$ ) representing the constraint to be satisfied.

In the rest of this paper, we simply call finite linear CSP as CSP.

We extend the functions  $\ell$  and  $u$  for any linear expressions  $e \in E(V)$ , e.g.  $\ell(2x - 3y) = -9$  and  $u(2x - 3y) = 6$  when  $\ell(x) = \ell(y) = 0$  and  $u(x) = u(y) = 3$ .

An *assignment* of a CSP  $(V, \ell, u, B, S)$  is a pair  $(\alpha, \beta)$  where  $\alpha$  is a mapping from  $V$  to  $\mathbf{Z}$  and  $\beta$  is a mapping from  $B$  to  $\{\top, \perp\}$ .

**Definition 2 (Satisfiability).** Let  $(V, \ell, u, B, S)$  be a CSP. A clause  $C \in C(V, B)$  is *satisfiable* by an assignment  $(\alpha, \beta)$  if the assignment makes the clause  $C$  be true and  $\ell(x) \leq \alpha(x) \leq u(x)$  for all  $x \in V$ . We denote this satisfiability relation as follows.

$$(\alpha, \beta) \models C$$

A clause  $C$  is satisfiable if  $C$  is satisfiable by some assignment.

A set of clauses is satisfiable when all clauses in the set are satisfiable by the same assignment. A logical formula is satisfiable when its clausal form is satisfiable. The CSP is satisfiable if  $S$  is satisfiable.

Finally, we define SAT as a special form of CSP.

**Definition 3 (SAT).** A SAT (Boolean Satisfiability Testing Problem) is a CSP without integer variables, that is,  $(\emptyset, \emptyset, \emptyset, B, S)$ .

## 3 Encoding finite linear CSP to SAT

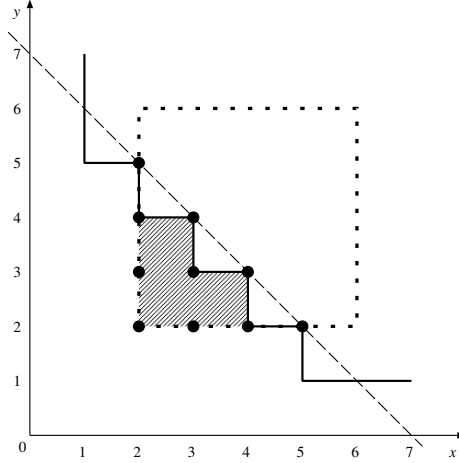
### 3.1 Converting comparisons to primitive comparisons

In this section, we will explain a method to transform a comparison into primitive comparisons.

A *primitive comparison* is a comparison in the form of  $x \leq c$  where  $x$  is an integer variable and  $c$  is an integer satisfying  $\ell(x) - 1 \leq c \leq u(x)$ . In fact, it is possible to restrict the range of  $c$  to  $\ell(x) \leq c \leq u(x) - 1$  since  $x \leq \ell(x) - 1$  is always false and  $x \leq u(x)$  is always true. However, we use the wider range to simplify the discussion.

Let us consider a comparison of  $x + y \leq 7$  when  $\ell(x) = \ell(y) = 0$  and  $u(x) = u(y) = 6$ . As shown in Figure 1, the comparison can be equivalently expressed as  $(x \leq 1 \vee y \leq 5) \wedge (x \leq 2 \vee y \leq 4) \wedge (x \leq 3 \vee y \leq 3) \wedge (x \leq 4 \vee y \leq 2) \wedge (x \leq 5 \vee y \leq 1)$  in which 10 black dotted points are contained as satisfiable assignments since  $0 \leq x, y \leq 6$ . Please note that conditions  $(x \leq 1 \vee y \leq 5)$  and  $(x \leq 5 \vee y \leq 1)$ , which are equivalent to  $y \leq 5$  and  $x \leq 5$  respectively, are necessary to exclude cases of  $x = 2, y = 6$  and  $x = 6, y = 2$ .

Now, we will show the following lemma before describing the conversion to primitive comparisons in general.



**Fig. 1.** Converting  $x + y \leq 7$  to primitive comparisons

**Lemma 1.** Let  $(V, \ell, u, B, S)$  be a CSP, then for any assignment  $(\alpha, \beta)$  of the CSP, for any linear expressions  $e, f \in E(V)$ , and for any integer  $c \geq \ell(e) + \ell(f)$ , the following holds.

$$\begin{aligned}
 & (\alpha, \beta) \models e + f \leq c \\
 \iff & (\alpha, \beta) \models \bigwedge_{a+b=c-1} (e \leq a \vee f \leq b)
 \end{aligned}$$

Parameters  $a$  and  $b$  range over  $\mathbf{Z}$  satisfying  $a + b = c - 1$ ,  $\ell(e) - 1 \leq a \leq u(e)$ , and  $\ell(f) - 1 \leq b \leq u(f)$ . The conjunction represents  $\top$  if there are no such  $a$  and  $b$ .

*Proof.* ( $\implies$ ) From the hypotheses and the definition of satisfiability, we get  $\alpha(e) + \alpha(f) \leq c$ ,  $\ell(e) \leq \alpha(e) \leq u(e)$ , and  $\ell(f) \leq \alpha(f) \leq u(f)$ . Let  $a$  and  $b$  be any integers satisfying  $a + b = c - 1$ ,  $\ell(e) - 1 \leq a \leq u(e)$ , and  $\ell(f) - 1 \leq b \leq u(f)$ . If there are no such  $a$  and  $b$ , the conclusion holds.

If  $\alpha(e) \leq a$ ,  $e \leq a$  in the conclusion is satisfied. Otherwise,  $f \leq b$  in the conclusion is satisfied since  $\alpha(f) \leq c - \alpha(e) \leq c - a - 1 = (a + b + 1) - a - 1 = b$ . Therefore,  $e \leq a \vee f \leq b$  is satisfied for any  $a$  and  $b$ .

( $\impliedby$ ) From the hypotheses,  $\alpha(e) \leq a \vee \alpha(f) \leq b$  is true for any  $a$  and  $b$  satisfying  $a + b = c - 1$ ,  $\ell(e) - 1 \leq a \leq u(e)$ , and  $\ell(f) - 1 \leq b \leq u(f)$ . From the definition of satisfiability, we also have  $\ell(e) \leq \alpha(e) \leq u(e)$  and  $\ell(f) \leq \alpha(f) \leq u(f)$ . Now, we show the conclusion through a proof by contradiction. Assume that  $\alpha(e) + \alpha(f) > c$  which is the negation of the conclusion.

When  $\alpha(e) \geq c - \ell(f) + 1$ , we choose  $a = c - \ell(f)$  and  $b = \ell(f) - 1$ . It is easy to check the conditions  $\ell(e) - 1 \leq a \leq u(e)$  and  $\ell(f) - 1 \leq b \leq u(f)$  are satisfied, and  $\alpha(e) \leq a \vee \alpha(f) \leq b$  becomes false for such  $a$  and  $b$ , which contradicts the hypotheses.

When  $\alpha(e) < c - \ell(f) + 1$ , we choose  $a = \alpha(e) - 1$  and  $b = c - \alpha(e)$ . It is easy to check the conditions  $\ell(e) - 1 \leq a \leq u(e)$  and  $\ell(f) - 1 \leq b \leq u(f)$  are satisfied, and  $\alpha(e) \leq a \vee \alpha(f) \leq b$  becomes false for such  $a$  and  $b$ , which contradicts the hypotheses.  $\square$

The following proposition shows a general method to convert a (linear) comparison into primitive comparisons.

**Proposition 1.** Let  $(V, \ell, u, B, S)$  be a CSP, then for any assignment  $(\alpha, \beta)$  of the CSP, for any linear expression  $\sum_{i=1}^n a_i x_i \in E(V)$ , and for any integer  $c \geq \ell(\sum_{i=1}^n a_i x_i)$  the following holds.

$$\begin{aligned} (\alpha, \beta) \models \sum_{i=1}^n a_i x_i \leq c \\ \iff (\alpha, \beta) \models \bigwedge_{\sum_{i=1}^n b_i = c - n + 1} \bigvee_i (a_i x_i \leq b_i)^\# \end{aligned}$$

Parameters  $b_i$ 's range over  $\mathbf{Z}$  satisfying  $\sum_{i=1}^n b_i = c - n + 1$  and  $\ell(a_i x_i) - 1 \leq b_i \leq u(a_i x_i)$  for all  $i$ . The translation  $(\ )^\#$  is defined as follows.

$$(a x \leq b)^\# \equiv \begin{cases} x \leq \left\lfloor \frac{b}{a} \right\rfloor & (a > 0) \\ \neg \left( x \leq \left\lfloor \frac{b}{a} \right\rfloor - 1 \right) & (a < 0) \end{cases}$$

*Proof.* The satisfiability of  $\sum a_i x_i \leq c$  is equivalent to the satisfiability of  $\bigwedge \bigvee (a_i x_i \leq b_i)$  from Lemma 1, and the satisfiability of each  $a_i x_i \leq b_i$  is equivalent to the satisfiability of  $(a_i x_i \leq b_i)^\#$ .  $\square$

Therefore, any comparison literal  $\sum a_i x_i \leq c$  in a CSP can be converted to a CNF (product-of-sums) formula of primitive comparisons (or Boolean constants) without changing its satisfiability. Please note that the comparison literal should occur positively in the CSP to perform this conversion.

*Example 1.* When  $\ell(x) = \ell(y) = \ell(z) = 0$  and  $u(x) = u(y) = u(z) = 3$ , comparison  $x + y < z - 1$  is converted into  $(x \leq -1 \vee y \leq -1 \vee \neg(z \leq 1)) \wedge (x \leq -1 \vee y \leq 0 \vee \neg(z \leq 2)) \wedge (x \leq -1 \vee y \leq 1 \vee \neg(z \leq 3)) \wedge (x \leq 0 \vee y \leq -1 \vee \neg(z \leq 2)) \wedge (x \leq 0 \vee y \leq 0 \vee \neg(z \leq 3)) \wedge (x \leq 1 \vee y \leq -1 \vee \neg(z \leq 3))$ .

### 3.2 Encoding to SAT

As shown in the previous subsection, any (finite linear) CSP can be converted into a CSP with only primitive comparisons.

Now, we eliminate each primitive comparison  $x \leq c$  ( $x \in V$ ,  $\ell(x) - 1 \leq c \leq u(x)$ ) by replacing it with a newly introduced Boolean variable  $p(x, c)$  which is chosen from  $\mathcal{B}$ . We denote a set of these new Boolean variables as follows.

$$B' = \{p(x, c) \mid x \in V, \ell(x) - 1 \leq c \leq u(x)\}$$

We also need to introduce the following axiom clauses  $A(x)$  for each integer variable  $x$  in order to represent the bound and the order relation.

$$A(x) = \{\{\neg p(x, \ell(x) - 1)\}, \{p(x, u(x))\}\} \\ \cup \{\{\neg p(x, c - 1), p(x, c)\} \mid \ell(x) \leq c \leq u(x)\}$$

As previously described, clauses of  $\{\neg p(x, \ell(x) - 1)\}$  and  $\{p(x, u(x))\}$  are redundant. However, these will be removed in the early stage of SAT solving and will not much affect the performance of the solver.

**Proposition 2.** Let  $(V, \ell, u, B, S)$  be a CSP with only primitive comparisons, let  $S^*$  be a clausal form formula obtained from  $S$  by replacing each primitive comparison  $x \leq c$  with  $p(x, c)$ , and let  $A = \bigcup_{x \in V} A(x)$ . Then, the following holds.

$$(V, \ell, u, B, S) \text{ is satisfiable} \\ \iff (\emptyset, \emptyset, \emptyset, B \cup B', S^* \cup A) \text{ is satisfiable}$$

*Proof.* ( $\implies$ ) Since  $(V, \ell, u, B, S)$  is satisfiable, there is an assignment  $(\alpha, \beta)$  which makes  $S$  be true and  $\ell(x) \leq \alpha(x) \leq u(x)$  for all  $x \in V$ . We extend the mapping  $\beta$  to  $\beta^*$  as follows.

$$\beta^*(p) = \begin{cases} \beta(p) & (p \in B) \\ \alpha(x) \leq c & (p = p(x, c) \in B') \end{cases}$$

Then an assignment  $(\alpha, \beta^*)$  satisfies  $S^* \cup A$ .

( $\impliedby$ ) From the hypotheses, there is an assignment  $(\emptyset, \beta)$  which makes  $S^* \cup A$  be true. We define a mapping  $\alpha$  as follows.

$$\alpha(x) = \min \{c \mid \ell(x) \leq c \leq u(x), p(x, c)\}$$

It is straightforward to check the assignment  $(\alpha, \beta)$  satisfies  $S$ . □

### 3.3 Keeping Clausal Form

When encoding a clause of CSP to SAT, the encoded formula is no more a clausal form in general.

Consider a case of encoding a clause  $\{x - y \leq -1, -x + y \leq -1\}$  which means  $x \neq y$ . Each of  $x - y \leq -1$  and  $-x + y \leq -1$  is encoded into a CNF formula of primitive comparisons. Therefore, when we expand the conjunctions to get a clausal form, the number of obtained clauses is the multiplication of two numbers of primitive comparisons.

As it is well known, introduction of new Boolean variables is useful to reduce the size. Suppose  $\{c_1, c_2, \dots, c_n\}$  is a clause of original CSP where  $c_i$ 's are comparison literals, and  $\{C_{i1}, C_{i2}, \dots, C_{in_i}\}$  is an encoded CNF formula (in clausal form) of  $c_i$  for each  $i$ .

We introduce new Boolean variables  $p_1, p_2, \dots, p_n$  chosen from  $\mathcal{B}$ , and replace the original clause with  $\{p_1, p_2, \dots, p_n\}$ . We also introduce new clauses  $\{\neg p_i\} \cup C_{ij}$  for each  $1 \leq i \leq n$  and  $1 \leq j \leq n_i$ .

This conversion does not affect the satisfiability which can be shown from the following Lemma.

**Lemma 2.** Let  $(V, \ell, u, B, S)$  be a CSP,  $\{L_1, L_2, \dots, L_n\}$  be a clause of the CSP, and  $p_1, p_2, \dots, p_n$  be new Boolean variables. Then, the following holds.

$$\begin{aligned} & \{L_1, L_2, \dots, L_n\} \text{ is satisfiable} \\ \iff & \{p_1, p_2, \dots, p_n\} \{ \neg p_1, L_1 \}, \{ \neg p_2, L_2 \}, \dots, \{ \neg p_n, L_n \} \text{ is satisfiable} \end{aligned}$$

*Proof.* ( $\implies$ ) From the hypotheses, there is an assignment  $(\alpha, \beta)$  which satisfies some  $L_i$ . We extend the mapping  $\beta$  so that  $\beta(p_i) = \top$  and  $\beta(p_j) = \perp$  ( $j \neq i$ ). Then, the assignment satisfies converted clauses.

( $\impliedby$ ) From the hypotheses, there is an assignment  $(\alpha, \beta)$  which satisfies some  $p_i$ . The assignment also satisfies  $\{\neg p_i, L_i\}$ , and therefore  $L_i$ . Hence the conclusion holds.  $\square$

*Example 2.* Consider an example of encoding a clause  $\{x - y \leq -1, -x + y \leq -1\}$  when  $\ell(x) = \ell(y) = 0$  and  $u(x) = u(y) = 2$ .  $x - y \leq -1$  and  $-x + y \leq -1$  are converted into  $S_1 = (p(x, -1) \vee \neg p(y, 0)) \wedge (p(x, 0) \vee \neg p(y, 1)) \wedge (p(x, 1) \vee \neg p(y, 2))$  and  $S_2 = (\neg p(x, 2) \vee p(y, 1)) \wedge (\neg p(x, 1) \vee p(y, 0)) \wedge (\neg p(x, 0) \vee p(y, -1))$  respectively. Expanding  $S_1 \vee S_2$  generates 9 clauses. However, by introducing new Boolean variables  $p$  and  $q$ , we obtain the following seven clauses.

$$\begin{array}{ccc} & \{p, q\} & \\ \{\neg p, p(x, -1), \neg p(y, 0)\} & \{\neg p, p(x, 0), \neg p(y, 1)\} & \{\neg p, p(x, 1), \neg p(y, 2)\} \\ \{\neg q, \neg p(x, 2), p(y, 1)\} & \{\neg q, \neg p(x, 1), p(y, 0)\} & \{\neg q, \neg p(x, 0), p(y, -1)\} \end{array}$$

### 3.4 Size of the Encoded SAT Problem

Usually the size of the encoded SAT problem becomes large.

Suppose the number of integer variables is  $n$ , and the size of integer variable domains is  $d$ , that is,  $d = u(x) - \ell(x) + 1$  for all  $x \in V$ . Then the size of newly introduced Boolean variables  $B'$  is  $O(nd)$ , the size of axiom clauses  $A$  is also  $O(nd)$ , and the number of literals in each axiom clause is at most two.

Each comparison  $\sum_{i=1}^m a_i x_i \leq c$  will be encoded into  $O(d^{m-1})$  clauses in general by Proposition 1.

However, it is possible to reduce the number of integer variables in each comparison at most three. For example,  $x_1 + x_2 + x_3 + x_4 \leq c$  can be replaced with  $x + x_3 + x_4 \leq c$  by introducing a new integer variable  $x$  and new constraints  $x - x_1 - x_2 \leq 0$  and  $-x + x_1 + x_2 \leq 0$ , that is,  $x = x_1 + x_2$ .

Therefore, each comparison  $\sum_{i=1}^m a_i x_i \leq c$  can be encoded by at most  $O(d^2) + O(md)$  clauses even when  $m \geq 4$ , and the number of literals in each clause is at most four (three for integer variables and one for the case handling described in the previous subsection).

$$(p_{ij}) = \begin{pmatrix} 661 & 6 & 333 \\ 168 & 489 & 343 \\ 171 & 505 & 324 \end{pmatrix}$$

**Fig. 2.** OSS benchmark instance `gp03-01`

## 4 Encoding finite linear COP to SAT

**Definition 4 (Finite linear COP).** A (finite linear) COP (Constraint Optimization Problem) is defined as a tuple  $(V, \ell, u, B, S, v)$  where

- (1)  $(V, \ell, u, B, S)$  is a finite linear CSP, and
- (2)  $v \in V$  is an integer variable representing the objective variable to be minimized (without loss of generality we assume COPs as minimization problems).

The optimal value of COP  $(V, \ell, u, B, S, v)$  can be obtained by repeatedly solving CSPs.

$$\min \{c \mid \ell(v) \leq c \leq u(v), \text{ CSP } (V, \ell, u, B, S \cup \{v \leq c\}) \text{ is satisfiable}\}$$

Of course, instead of linear search, binary search method is useful to find the optimal value efficiently as used in previous works [10–12].

It is also possible to encode COP to SAT once at first, and repeatedly modify only the clause  $\{v \leq c\}$  for a given  $c$ . This procedure substantially reduces the time spent for encoding.

## 5 Solving OSS

In order to show the applicability of our method, we applied it to OSS (Open-Shop Scheduling) problems. There are three well-known sets of OSS benchmark problems by Guéret and Prins [15] (80 instances denoted by `gp*`), Taillard [16] (60 instances denoted by `tai_*`), and Brucker et al. [17] (52 instances denoted by `j*`), which are also used in [18–20].

Some problems in these benchmark sets are very hard to solve. Actually, three instances (`j7-per0-0`, `j8-per0-1`, and `j8-per10-2`) are still open, and 37 instances are closed recently in 2005 by complete MCS-based search solver of ILOG [20].

Representing OSS problem as CSP is straightforward. Figure 2 defines a benchmark instance `gp03-01` of 3 jobs and 3 machines. Each element  $p_{ij}$  represents the process time of the operation  $O_{ij}$  ( $0 \leq i, j \leq 2$ ). The instance `gp03-01` can be represented as a CSP of 27 clauses as shown in Figure 3.

In the figure, integer variables  $m$  represents the makespan and each  $s_{ij}$  represents the start time of each operation  $O_{ij}$ . Clauses  $\{s_{ij} + p_{ij} \leq m\}$  represent



$$\begin{array}{lll}
\{s_{00} + 661 \leq m\} & \{s_{01} + 6 \leq m\} & \{s_{02} + 333 \leq m\} \\
\{s_{10} + 168 \leq m\} & \{s_{11} + 489 \leq m\} & \{s_{12} + 343 \leq m\} \\
\{s_{20} + 171 \leq m\} & \{s_{21} + 505 \leq m\} & \{s_{22} + 324 \leq m\} \\
\{s_{00} + 661 \leq s_{01}, s_{01} + 6 \leq s_{00}\} & \{s_{00} + 661 \leq s_{02}, s_{02} + 333 \leq s_{00}\} & \\
\{s_{01} + 6 \leq s_{02}, s_{02} + 333 \leq s_{01}\} & \{s_{10} + 168 \leq s_{11}, s_{11} + 489 \leq s_{10}\} & \\
\{s_{10} + 168 \leq s_{12}, s_{12} + 343 \leq s_{10}\} & \{s_{11} + 489 \leq s_{12}, s_{12} + 343 \leq s_{11}\} & \\
\{s_{20} + 171 \leq s_{21}, s_{21} + 505 \leq s_{20}\} & \{s_{20} + 171 \leq s_{22}, s_{22} + 324 \leq s_{20}\} & \\
\{s_{21} + 505 \leq s_{22}, s_{22} + 324 \leq s_{21}\} & \{s_{00} + 661 \leq s_{10}, s_{10} + 168 \leq s_{00}\} & \\
\{s_{00} + 661 \leq s_{20}, s_{20} + 171 \leq s_{00}\} & \{s_{10} + 168 \leq s_{20}, s_{20} + 171 \leq s_{10}\} & \\
\{s_{01} + 6 \leq s_{11}, s_{11} + 489 \leq s_{01}\} & \{s_{01} + 6 \leq s_{21}, s_{21} + 505 \leq s_{01}\} & \\
\{s_{11} + 489 \leq s_{21}, s_{21} + 505 \leq s_{11}\} & \{s_{02} + 333 \leq s_{12}, s_{12} + 343 \leq s_{02}\} & \\
\{s_{02} + 333 \leq s_{22}, s_{22} + 324 \leq s_{02}\} & \{s_{12} + 343 \leq s_{22}, s_{22} + 324 \leq s_{12}\} & 
\end{array}$$

**Fig. 3.** CSP representation of gp03-01

deadline constraint such that operations should be completed before  $m$ . Clauses  $\{s_{ij} + p_{ij} \leq s_{kl}, s_{kl} + p_{kl} \leq s_{ij}\}$  represent resource capacity constraint such that the operation  $O_{ij}$  and  $O_{kl}$  should not be overlapped each other.

Before encoding the CSP to SAT, we also need to determine the lower and upper bound of integer variables. We used the following values  $\ell$  and  $u$  (where  $n$  is the number of jobs and machines).

$$\begin{aligned}
\ell &= \max \left( \max_{0 \leq i < n} \sum_{0 \leq j < n} p_{ij}, \max_{0 \leq j < n} \sum_{0 \leq i < n} p_{ij} \right) \\
u &= \sum_{0 \leq k < n} \max_{(i-j) \bmod n = k} p_{ij}
\end{aligned}$$

The value  $u$  is used for the upper bound of  $s_{ij}$ 's and  $m$ , and the value  $\ell$  is used for the lower bound of  $m$  (the lower bound 0 is used for  $s_{ij}$ 's). For example,  $\ell = 1000$  and  $u = 1509$  for the instance gp03-01.

We developed a program called **CSP2SAT** which encodes a CSP representation (of a given OSS problem) into SAT and repeatedly invokes a complete SAT solver to find the optimal solution by binary search<sup>3</sup>. We used MiniSat [5] as the backend complete SAT solver because it is known to be very efficient (MiniSat is a winner of all industrial categories of the SAT 2005 competition).

We run **CSP2SAT** for all 192 instances of the three benchmark sets on Intel Xeon 2.8GHz 4GB memory machine with the time limit of 3 hours (10800 seconds).

Figures 7, 8, and 9 provides the results. The column named ‘‘Optim.’’ describes the optimal value found by the program, and ‘‘CPU’’ describes the total CPU time in seconds including encoding process. The column named ‘‘SAT’’ describes the numbers of Boolean variables and clauses in the encoded SAT

<sup>3</sup> The program will be available at <http://bach.istc.kobe-u.ac.jp/csp2sat/>.

$$(s_{ij}) = \begin{pmatrix} 247 & 296 & 110 & 618 & 537 & 31 & 500 & 127 \\ 815 & 50 & 328 & 274 & 311 & 672 & 550 & 6 \\ 1 & 583 & 120 & 339 & 876 & 842 & 675 & 58 \\ 293 & 669 & 5 & 72 & 250 & 502 & 403 & 994 \\ 286 & 517 & 870 & 594 & 612 & 347 & 0 & 297 \\ 404 & 252 & 73 & 28 & 83 & 25 & 300 & 734 \\ 707 & 997 & 560 & 12 & 48 & 87 & 842 & 340 \\ 53 & 6 & 703 & 285 & 342 & 872 & 526 & 547 \end{pmatrix}$$

**Fig. 4.** Optimal Scheduling of `j8-per10-2` found by CSP2SAT

problem. Although time spent for encoding is not shown separately in the figures, it ranges from 1 second to 1163 seconds and fits linearly with the number of clauses in the encoded SAT program.

CSP2SAT found the optimal solutions for 189 known problems and one unknown problem (`j8-per10-2`) within 3 hours.

The known upper bound of `j8-per10-2` was 1009. CSP2SAT improved the result to 1002 and proved there are no solutions for 1001. Figure 4 shows the start times  $s_{ij}$  of the optimal scheduling found by the program.

Figure 5 provides the log scale plot of the number of clauses in the encoded SAT problem ( $x$ -axis) and the total CPU time ( $y$ -axis) for 190 problems. The mark  $+$  is used for `gp*` benchmarks,  $\times$  is used for `tai*` benchmarks, and  $\diamond$  is used for `j*` benchmarks. Dotted line is a plot of  $y = 0.00006x$ .

Except some instances of `j*` benchmarks, it seems the total CPU time linearly fits with the number of clauses. This shows that the encoding used in this paper is natural and does not uselessly increase the complexity for SAT solver.

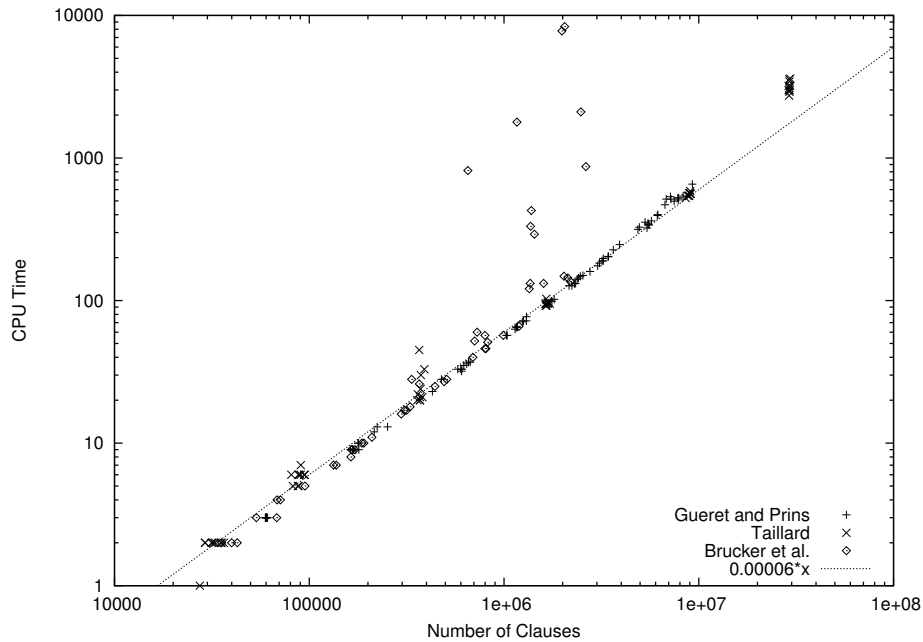
For the remaining two open problems `j7-per0-0` and `j8-per0-1`, we solved and proved their optimal values by using 10 Mac mini machines (PowerPC G4 1.42GHz 1GB memory) running in parallel on Xgrid system [21] and by dividing the problem into 120 subproblems where each subproblem is obtained by specifying the order of six operations. Optimal solutions were found and proved for both of the two remaining instances within 13 hours.

Figure 6 summarizes the newly obtained results. All three remaining open problems in [18–20] are now closed.

## 6 Conclusion

In this paper, we proposed a method to encode Constraint Satisfaction Problems (CSP) and Constraint Optimization Problems (COP) with integer linear constraints into Boolean Satisfiability Testing Problems (SAT).

To evaluate the effectiveness of the encoding, we applied the method to Open-Shop Scheduling Problems (OSS). All 192 instances in three OSS benchmark sets are examined, and our program found and proved the optimal results for all instances including three previously undecided problems.



**Fig. 5.** Log scale plot of the number of clauses and the CPU time

Instance	Makespan	Previously known bounds	
		Lower bound	Upper bound
j7-per0-0	1048	1039	1048
j8-per0-1	1039	1018	1039
j8-per10-2	<b>1002</b>	1000	1009

**Fig. 6.** New results found and proved to be optimal

## Acknowledgments

We would like to give thanks to Katsumi Inoue, Hidetomo Nabeshima, Takehide Soh, and Shuji Ohnishi for their helpful suggestions.

## References

1. Selman, B., Kautz, H.A., Cohen, B.: Local search strategies for satisfiability testing. DIMACS Series in Discrete Mathematics and Theoretical Computer Science **26** (1996) 521–532
2. Li, C.M., Anbulagan: Heuristics based on unit propagation for satisfiability problems. In: Proceedings of the Fifteenth International Joint Conference on Artificial Intelligence (IJCAI 97). (1997) 366–371
3. Marques-Silva, J.P., Sakallah, K.A.: GRAPS: A search algorithm for propositional satisfiability. IEEE Trans. Computers **48** (1999) 506–521

Instance	Optim.	CPU	SAT		Instance	Optim.	CPU	SAT	
			Variables	Clauses				Variables	Clauses
gp03-01	1168	3	14155	61133	gp07-01	1159	99	137537	1761090
gp03-02	1170	3	13945	59978	gp07-02	1185	148	188537	2461830
gp03-03	1168	3	13945	59978	gp07-03	1237	132	179037	2331300
gp03-04	1166	3	13995	60253	gp07-04	1167	131	176437	2295576
gp03-05	1170	3	13855	59483	gp07-05	1157	141	182137	2373894
gp03-06	1169	3	13915	59813	gp07-06	1193	127	166587	2160237
gp03-07	1165	3	13925	59868	gp07-07	1185	102	141187	1811241
gp03-08	1167	3	13955	60033	gp07-08	1180	144	184787	2410305
gp03-09	1162	3	14075	60693	gp07-09	1220	150	194437	2542896
gp03-10	1165	3	13945	59978	gp07-10	1270	127	171837	2323272
gp04-01	1281	10	28097	179010	gp08-01	1130	160	186315	2762188
gp04-02	1270	13	33928	223257	gp08-02	1135	190	216215	3233688
gp04-03	1288	9	28182	179655	gp08-03	1110	197	215955	3229588
gp04-04	1261	12	32925	215646	gp08-04	1153	227	242020	3640613
gp04-05	1289	10	27927	177720	gp08-05	1218	247	259830	3921463
gp04-06	1269	9	27383	173592	gp08-06	1115	175	203085	3026638
gp04-07	1267	9	25955	162756	gp08-07	1126	204	229215	3438688
gp04-08	1259	9	26516	167013	gp08-08	1148	183	207245	3092238
gp04-09	1280	9	26737	168690	gp08-09	1114	189	213225	3186538
gp04-10	1263	13	37736	252153	gp08-10	1161	203	227980	3419213
gp05-01	1245	36	72727	643703	gp09-01	1129	323	317881	5423978
gp05-02	1247	33	65993	578694	gp09-02	1110	327	291477	4954180
gp05-03	1265	37	75457	670058	gp09-03	1115	395	357077	6121380
gp05-04	1258	23	50497	429098	gp09-04	1130	340	322063	5498387
gp05-05	1280	33	68151	599527	gp09-05	1180	362	333871	5708483
gp05-06	1269	37	74131	657257	gp09-06	1093	401	359455	6163691
gp05-07	1269	32	68801	605802	gp09-07	1090	339	325507	5559665
gp05-08	1287	28	55489	477290	gp09-08	1105	349	321325	5485256
gp05-09	1262	35	70387	621113	gp09-09	1123	316	286803	4871017
gp05-10	1254	33	69009	607810	gp09-10	1110	355	310993	5301422
gp06-01	1264	57	96410	1038543	gp10-01	1093	470	353491	6705492
gp06-02	1285	65	106659	1158484	gp10-02	1097	526	412677	7878078
gp06-03	1255	72	115317	1259806	gp10-03	1081	535	376317	7157718
gp06-04	1275	63	104957	1138566	gp10-04	1077	515	378438	7199739
gp06-05	1299	65	107806	1171907	gp10-05	1071	515	358743	6809544
gp06-06	1284	65	106400	1155453	gp10-06	1071	508	410960	7844061
gp06-07	1290	77	119091	1303972	gp10-07	1079	523	408839	7802040
gp06-08	1265	71	113726	1241187	gp10-08	1093	498	392578	7479879
gp06-09	1243	72	118943	1302240	gp10-09	1112	541	434897	8318298
gp06-10	1254	57	95559	1028584	gp10-10	1092	656	483276	9276777

Fig. 7. Results for benchmark instances provided by Guéret and Prins

4. Moskewicz, M.W., Madigan, C.F., Zhao, Y., Zhang, L., Malik, S.: Chaff: Engineering an efficient SAT solver. In: Proceedings of the 38th Design Automation Conference (DAC 2001). (2001) 530–535
5. Eén, N., Sörensson, N.: An extensible SAT-solver. In: Proceedings of the 6th International Conference on Theory and Applications of Satisfiability Testing (SAT 2003). (2003) 502–518
6. Kautz, H.A., McAllester, D.A., Selman, B.: Encoding plans in propositional logic. In: Proceedings of the 5th International Conference on Principles of Knowledge Representation and Reasoning (KR'96). (1996) 374–384
7. Ernst, M.D., Millstein, T.D., Weld, D.S.: Automatic SAT-compilation of planning problems. In: Proceedings of the 15th International Joint Conference on Artificial Intelligence (IJCAI 97). (1997) 1169–1177
8. Hoos, H.H.: SAT-encodings, search space structure, and local search performance. In: Proceedings of the 16th International Joint Conference on Artificial Intelligence (IJCAI 99). (1999) 296–303

Instance	Optim. CPU	SAT		Instance	Optim. CPU	SAT			
		Variables	Clauses			Variables	Clauses		
tai_4x4.1	193	2	5043	31706	tai_10x10.1	637	98	94183	1678890
tai_4x4.2	236	1	4643	27426	tai_10x10.2	588	95	95343	1716326
tai_4x4.3	271	2	5460	32925	tai_10x10.3	598	92	92303	1651992
tai_4x4.4	250	2	5358	32341	tai_10x10.4	577	92	91314	1639647
tai_4x4.5	295	2	6081	36418	tai_10x10.5	640	96	93978	1677177
tai_4x4.6	189	2	4721	29194	tai_10x10.6	538	95	91151	1642608
tai_4x4.7	201	2	4743	29188	tai_10x10.7	616	103	92285	1648788
tai_4x4.8	217	2	5629	35110	tai_10x10.8	595	95	91094	1631685
tai_4x4.9	261	2	5328	31517	tai_10x10.9	595	97	94528	1697235
tai_4x4.10	217	2	5611	35444	tai_10x10.10	596	95	93315	1674220
tai_5x5.1	300	6	11526	94098	tai_15x15.1	937	523	309784	8563684
tai_5x5.2	262	5	10110	82314	tai_15x15.2	918	567	325397	9026993
tai_5x5.3	323	6	11318	90297	tai_15x15.3	871	543	315726	8767426
tai_5x5.4	310	5	11047	88190	tai_15x15.4	934	560	326511	9067128
tai_5x5.5	326	6	10356	80906	tai_15x15.5	946	541	323109	8940331
tai_5x5.6	312	5	10942	87344	tai_15x15.6	933	560	326512	9067214
tai_5x5.7	303	6	10951	87906	tai_15x15.7	891	566	322034	8943618
tai_5x5.8	300	6	11009	88852	tai_15x15.8	893	546	319320	8866998
tai_5x5.9	353	6	11940	94884	tai_15x15.9	899	568	324060	8998985
tai_5x5.10	326	7	11344	90508	tai_15x15.10	902	586	325865	9053491
tai_7x7.1	435	21	30952	370295	tai_20x20.1	1155	3105	775142	29178719
tai_7x7.2	443	24	31244	372853	tai_20x20.2	1241	3559	777061	29153596
tai_7x7.3	468	30	31669	374258	tai_20x20.3	1257	2990	770228	28898989
tai_7x7.4	463	20	31224	370305	tai_20x20.4	1248	3442	779059	29238508
tai_7x7.5	416	22	30171	360661	tai_20x20.5	1256	3603	785066	29485803
tai_7x7.6	451	45	30986	367026	tai_20x20.6	1204	2741	773489	29073596
tai_7x7.7	422	33	32415	389596	tai_20x20.7	1294	2912	779414	29225385
tai_7x7.8	424	20	30863	370287	tai_20x20.8	1169	2990	778336	29262619
tai_7x7.9	458	21	31929	380761	tai_20x20.9	1289	3204	785835	29493666
tai_7x7.10	398	20	29939	359194	tai_20x20.10	1241	3208	770645	28917758

**Fig. 8.** Results for benchmark instances provided by Taillard

9. Crawford, J.M., Baker, A.B.: Experimental results on the application of satisfiability algorithms to scheduling problems. In: Proceedings of the 12th National Conference on Artificial Intelligence (AAAI-94). (1994) 1092–1097
10. Soh, T., Inoue, K., Banbara, M., Tamura, N.: Experimental results for solving job-shop scheduling problems with multiple SAT solvers. In: Proceedings of the 1st International Workshop on Distributed and Speculative Constraint Processing (DSCP'05). (2005)
11. Inoue, K., Soh, T., Ueda, S., Sasaura, Y., Banbara, M., Tamura, N.: A competitive and cooperative approach to propositional satisfiability. *Discrete Applied Mathematics* (2006) (to appear).
12. Nabeshima, H., Soh, T., Inoue, K., Iwanuma, K.: Lemma reusing for SAT based planning and scheduling. In: Proceedings of the International Conference on Automated Planning and Scheduling 2006 (ICAPS'06). (2006) 103–112
13. de Kleer, J.: A comparison of ATMS and CSP techniques. In: Proceedings of the 11th International Joint Conference on Artificial Intelligence (IJCAI 89). (1989) 290–296
14. Iwama, K., Miyazaki, S.: SAT-variable complexity of hard combinatorial problems. In: Proceedings of the IFIP 13th World Computer Congress. (1994) 253–258
15. Guéret, C., Prins, C.: A new lower bound for the open-shop problem. *Annals of Operations Research* **92** (1999) 165–183
16. Taillard, E.D.: Benchmarks for basic scheduling problems. *European Journal of Operational Research* **64** (1993) 278–285

Instance	Optim. CPU	SAT		Instance	Optim. CPU	SAT			
		Variables	Clauses			Variables	Clauses		
j3-per0-1	1127	2	10805	42708	j6-per0-0	1056	817	63443	652740
j3-per0-2	1084	5	20335	95123	j6-per0-1	1045	57	92340	990913
j3-per10-0	1131	3	12675	53453	j6-per0-2	1063	57	75801	797362
j3-per10-1	1069	3	15335	68062	j6-per10-0	1005	52	67661	705462
j3-per10-2	1053	4	15355	68341	j6-per10-1	1021	46	76467	808206
j3-per20-0	1026	2	10015	39923	j6-per10-2	1012	51	77799	823964
j3-per20-1	1000	2	9245	35496	j6-per20-0	1000	60	69400	727773
j3-per20-2	1000	4	15755	71137	j6-per20-1	1000	46	75431	798740
j4-per0-0	1055	7	22062	133215	j6-per20-2	1000	40	66181	692002
j4-per0-1	1180	11	32160	209841	j7-per0-0	-	-	85887	1051419
j4-per0-2	1071	8	26057	163530	j7-per0-1	1055	428	109837	1380492
j4-per10-0	1041	10	29457	190740	j7-per0-2	1056	292	113537	1431330
j4-per10-1	1019	7	22538	137589	j7-per10-0	1013	332	108687	1368170
j4-per10-2	1000	9	26057	164892	j7-per10-1	1000	121	107087	1347411
j4-per20-0	1000	10	28726	186429	j7-per10-2	1011	1786	93887	1165467
j4-per20-1	1004	9	26074	165849	j7-per20-0	1000	66	95487	1193523
j4-per20-2	1009	9	26822	171525	j7-per20-1	1005	132	125087	1595847
j5-per0-0	1042	28	40825	335726	j7-per20-2	1003	132	107987	1361349
j5-per0-1	1054	28	58687	508163	j8-per0-1	-	-	145495	2118473
j5-per0-2	1063	26	44127	367603	j8-per0-2	1052	870	177995	2630988
j5-per10-0	1004	18	39967	329523	j8-per10-0	1017	2107	168310	2481679
j5-per10-1	1002	17	37653	307928	j8-per10-1	1000	8346	140620	2047787
j5-per10-2	1006	16	36509	296700	j8-per10-2	1002	7789	136655	1984646
j5-per20-0	1000	17	38329	315830	j8-per20-0	1000	148	139255	2030756
j5-per20-1	1000	27	56607	492707	j8-per20-1	1000	136	149265	2191364
j5-per20-2	1012	25	51485	442196	j8-per20-2	1000	144	145300	2125157

**Fig. 9.** Results for benchmark instances provided by Brucker et al.

17. Brucker, P., Hurink, J., Jurisch, B., Wöstmann, B.: A branch & bound algorithm for the open-shop problem. *Discrete Applied Mathematics* **76** (1997) 43–59
18. Jussien, N., Lhomme, O.: Local search with constraint propagation and conflict-based heuristics. *Artificial Intelligence* **139** (2002) 21–45
19. Blum, C.: Beam-ACO — hybridizing ant colony optimization with beam search: an application to open shop scheduling. *Computers & OR* **32** (2005) 1565–1591
20. Laborie, P.: Complete MCS-based search: Application to resource constrained project scheduling. In: *Proceedings of the Nineteenth International Joint Conference on Artificial Intelligence (IJCAI-05)*. (2005) 181–186
21. Apple Computer Inc.: *Xgrid Guide*. (2004)